

The screening Horndeski cosmologies

Alexei A. Starobinsky,^{a,b} Sergey V. Sushkov,^b
Mikhail S. Volkov^{c,b,1}

^aL.D.Landau Institute for Theoretical Physics RAS, Moscow 119334, Russia

^bDepartment of General Relativity and Gravitation, Institute of Physics,
Kazan Federal University, Kremlevskaya street 18, 420008 Kazan, Russia

^cLaboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350,
Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail: alstar@landau.ac.ru, sergey_sushkov@mail.ru,
volkov@lmpt.univ-tours.fr

Abstract. We present a systematic analysis of homogeneous and isotropic cosmologies in a particular Horndeski model with Galileon shift symmetry, containing also a Λ -term and a matter. The model, sometimes called Fab Five, admits a rich spectrum of solutions. Some of them describe the standard late time cosmological dynamic dominated by the Λ -term and matter, while at the early times the universe expands with a constant Hubble rate determined by the value of the scalar kinetic coupling. For other solutions the Λ -term and matter are screened at all times but there are nevertheless the early and late accelerating phases. The model also admits bounces, as well as peculiar solutions describing “the emergence of time”. Most of these solutions contain ghosts in the scalar and tensor sectors. However, a careful analysis reveals three different branches of ghost-free solutions, all showing a late time acceleration phase. We analyse the dynamical stability of these solutions and find that all of them are stable in the future, since all their perturbations stay bounded at late times. However, they all turn out to be unstable in the past, as their perturbations grow violently when one approaches the initial spacetime singularity. We therefore conclude that the model has no viable solutions describing the whole of the cosmological history, although it may describe the current acceleration phase. We also check that the flat space solution is ghost-free in the model, but it may acquire ghost in more general versions of the Horndeski theory.

¹Corresponding author.

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1 Introduction – Horndeski theory

The discovery of the current universe acceleration [1, 2] requires a theoretical explanation. From the phenomenological viewpoint, a small cosmological term is a very good explanation [3], but it is problematic from the quantum field theory viewpoint, since it is difficult to explain the origin and value of this term [4]. Therefore, alternative dark matter models have been proposed, most of which introduce a scalar field, as in the Brans-Dicke, quintessence, k -essence, etc. theories (see [5, 6] for reviews), while the others, as for example the $F(R)$ gravity [7, 8], although looking different, are equivalent to the theory with a scalar field. Some of these models were actually introduced long ago in the context of the inflation theory [9], and some describe both the primordial inflation and the late time acceleration [10].

In view of this interest towards theories with a gravitating scalar field one may ask, what is the most general theory of this type? The answer was obtained already in 1974 by Horndeski [11] – this theory should have at most second order field equations to avoid the Ostrogradsky ghost [12], and it is determined by the following action density (in the parameterization of Ref.[13])

$$L_H = \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5), \quad (1.1)$$

where, with $X \equiv -\frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi$, one has

$$\begin{aligned} \mathcal{L}_2 &= G_2(X, \Phi), \\ \mathcal{L}_3 &= G_3(X, \Phi) \square \Phi, \\ \mathcal{L}_4 &= G_4(X, \Phi) R + \partial_X G_4(X, \Phi) \delta^{\mu\nu}_{\alpha\beta} \nabla_\mu^\alpha \Phi \nabla_\nu^\beta \Phi, \\ \mathcal{L}_5 &= G_5(X, \Phi) G_{\mu\nu} \nabla^{\mu\nu} \Phi - \frac{1}{6} \partial_X G_5(X, \Phi) \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \nabla_\mu^\alpha \Phi \nabla_\nu^\beta \Phi \nabla_\rho^\gamma \Phi, \end{aligned} \quad (1.2)$$

the coefficient functions $G_k(X, \Phi)$ can be arbitrary, also $\delta_{\nu\alpha}^{\lambda\rho} = 2! \delta_{[\nu}^{\lambda} \delta_{\alpha]}^{\rho}$ and $\delta_{\nu\alpha\beta}^{\lambda\rho\sigma} = 3! \delta_{[\nu}^{\lambda} \delta_{\alpha}^{\rho} \delta_{\beta]}^{\sigma}$. This theory contains all previously studied models with a gravity-coupled scalar field. Recently it was rediscovered in the context of the covariant Galileon models [14, 15] (yet more recently it was found that it can be further generalised to allow higher order derivatives in the field equations in such a way that the number of propagating degrees of freedom is still three [16–18], [19, 20]). Horndeski cosmologies were studied in Refs. [13], [21–24], [25–27].

The general Horndeski theory is difficult to analyse without specifying somehow the coefficient functions $G_k(X, \Phi)$. There is a special subclass of the theory, sometimes called Fab Four (F4) [28, 29], for which the coefficients are chosen such that the Lagrangian becomes

$$L_{\text{F4}} = \sqrt{-g} (\mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda) \quad (1.3)$$

with

$$\begin{aligned} \mathcal{L}_J &= V_J(\Phi) G_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi, \\ \mathcal{L}_P &= V_P(\Phi) P_{\mu\nu\rho\sigma} \nabla^\mu \Phi \nabla^\rho \Phi \nabla^{\nu\sigma} \Phi, \\ \mathcal{L}_G &= V_G(\Phi) R, \\ \mathcal{L}_R &= V_R(\Phi) (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2). \end{aligned} \quad (1.4)$$

Here the double dual of the Riemann tensor is

$$P^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{4} \delta_{\sigma\lambda\alpha\beta}^{\mu\nu\gamma\delta} R^{\sigma\lambda}{}_{\gamma\delta} = -R^{\mu\nu}{}_{\alpha\beta} + 2R_{[\alpha}^{\mu} \delta_{\beta]}^{\nu} - 2R_{[\alpha}^{\nu} \delta_{\beta]}^{\mu} - R \delta_{[\alpha}^{\mu} \delta_{\beta]}^{\nu}, \quad (1.5)$$

whose contraction is the Einstein tensor, $P^{\mu\alpha}{}_{\nu\alpha} = G^{\mu}{}_{\nu}$. This model is distinguished by the *screening property* – it is the most general subclass of the Horndeski theory in which flat space is a solution, despite the presence of the cosmological term Λ . This property suggests that Λ is actually irrelevant and hence there is no need to explain its value. Indeed, however large Λ is, Minkowski space is always a solution and so one may hope that a slowly accelerating universe will be a solution as well. Although Refs. [28, 29] did not explain how to produce a small value of the actual Hubble parameter, the idea apparently was that a possibility for this should exist and should not depend on Λ , since the latter can be screened altogether. The F4 cosmologies were studied in [30], and it was found that the coefficient functions V_J, \dots, V_R can be adjusted in such a way that the theory mimics all phases of the universe expansion.

A particular model related to the F4 theories has received a lot of attention. Setting the potential functions to constant values, $V_J = -\alpha$, $V_P = V_R = 0$, $V_G = M_{\text{Pl}}^2$, but adding an extra term – the standard kinetic term X for the scalar, gives a model sometimes called Fab Five (F5) [31],

$$S = \frac{1}{2} \int (M_{\text{Pl}}^2 R - (\alpha G_{\mu\nu} + \varepsilon g_{\mu\nu}) \nabla^\mu \Phi \nabla^\nu \Phi - 2\Lambda) \sqrt{-g} d^4x + S_{\text{m}} \equiv \frac{1}{2} \int L d^4x + S_{\text{m}}. \quad (1.6)$$

Here $M_{\text{Pl}} = \sqrt{1/8\pi G}$ is the Planck mass, ε is a parameter, and S_{m} describes an ordinary matter assumed to be a perfect fluid. This model can be integrated completely

in the static and spherically symmetric sector [32–35]. Moreover, for $\varepsilon \neq 0$ it admits solutions for which the Λ -term is totally screened as in the F4 theory, but the metric is not flat but rather de Sitter with the Hubble rate proportional to ε/α [36, 37], [31].

We find this model interesting because it offers an opportunity to describe the late time cosmic self-acceleration while screening the Λ -term and hence circumventing the cosmological constant problem. One should say that this model is certainly not the most general one in this respect, as there are other models that can also show the self-acceleration and screening. For example, these can be models obtained by adding the X kinetic term to the generic F4 (1.3). One can also directly modify the mini-superspace Lagrangian such that the theory admits a de Sitter solution while screening the Λ -term [38, 39]. However, we prefer to consider the model (1.6) because it is manifestly covariant, and also because it is simple enough to be integrated completely. At the same time, the results we obtain suggest that the model should probably be generalized to have more realistic solutions, but this can be achieved only at the sake of losing simplicity.

In what follows we systematically study the homogeneous and isotropic cosmologies in the F5 model (1.6) and we find a rich spectrum of solutions. Some of them describe the standard late time cosmological dynamic dominated by the Λ -term and matter, while at early times the universe expands with a constant Hubble rate determined by the value of the scalar kinetic coupling. For other solutions the Λ -term and matter are screened at all times but there are nevertheless the early and late accelerating stages. The model also admits bounces, as well as peculiar solutions describing a creation of universe “out of nothing”. Most of these solutions contain ghosts in the scalar and tensor sectors, but for $\varepsilon \geq 0$ and for $\alpha \geq 0$ there are ghost-free solutions. We find three different branches of such solutions, all showing a late time acceleration phase, and for a certain range of the parameter ε the late time Hubble rate being determined by the ratio ε/α and not by Λ . Therefore, the screening mechanism works indeed, and it is probably easier to explain a small value of ε/α rather than that of Λ . We also check that the flat space solution is ghost-free in the model, but it may acquire ghost within the full F4 theory.

We conclude that the model may indeed be successful, in particular at late times. However, it cannot apply at all times, since its radiation-dominated solution has ghost and should be excluded from consideration, whereas ghost-free solutions do not show a radiation-dominated phase since they screen the matter together with the Λ -term. Without this phase the model cannot correctly describe the primary nucleosynthesis. As a result, the screening works “too well” in the model.

We also analyse the dynamical stability of the ghost-free solutions and find that all of them are stable in the future, since all their perturbations stay bounded at late times. However, they all turn out to be unstable in the past, as their perturbations grow violently when one approaches the initial spacetime singularity. We therefore conclude that the F5 model has no viable solutions describing the whole of the cosmological history. However, since it admits stable in the future solutions, it may well describe the current acceleration phase, hence it fulfills the main motivation for considering models with scalar field. More realistic models may probably exist in more general

versions of the Horndeski theory.

The rest of the text is organised as follows. Equations describing homogeneous and isotropic cosmologies are derived in the next section. Solutions of these equations are constructed in Sec. III first in the early and late time limits and then globally. All solutions are ghost-checked and classified accordingly. Sec. IV contains the stability analysis of the ghost-free solutions and concluding remarks. Many technical details are given in the three Appendices – the derivation of the no-ghost conditions in Appendix A, the no-ghost conditions for flat space within the full F4 theory in Appendix B, and the equations for generic perturbations of spatially flat cosmologies in Appendix C.

2 Homogeneous and isotropic cosmologies

The first variation of the action (1.6) is

$$\delta S = \frac{1}{2} \int (E_{\mu\nu} \delta g^{\mu\nu} + E_\Phi \delta \Phi) \sqrt{-g} d^4 x, \quad (2.1)$$

whose vanishing implies the gravitational equations,

$$E_{\mu\nu} \equiv M_{\text{Pl}}^2 G_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha \mathcal{T}_{\mu\nu} - \varepsilon T_{\mu\nu}^{(\Phi)} - T_{\mu\nu}^{(\text{m})} = 0, \quad (2.2)$$

with

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= P_{\alpha\mu\nu\beta} \nabla^\alpha \Phi \nabla^\beta \Phi + \frac{1}{2} g_{\mu\lambda} \delta_{\nu\alpha\beta}^{\lambda\rho\sigma} \nabla_\rho^\alpha \Phi \nabla_\sigma^\beta \Phi - X G_{\mu\nu}, \\ T_{\mu\nu}^{(\Phi)} &= \nabla_\mu \Phi \nabla_\nu \Phi + X g_{\mu\nu}, \\ T_{\mu\nu}^{(\text{m})} &= (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \end{aligned} \quad (2.3)$$

and the scalar equation

$$E_\Phi \equiv \nabla_\mu ((\alpha G^{\mu\nu} + \varepsilon g^{\mu\nu}) \nabla_\nu \Phi) = 0. \quad (2.4)$$

This latter equation has the structure of current conservation, due to the theory invariance under shifts $\Phi \rightarrow \Phi + \Phi_0$. The structure of this equation also implies that the propagation of Φ is determined by the effective “optical” metric $\mathcal{M}_{\mu\nu} = \alpha G_{\mu\nu} + \varepsilon g_{\mu\nu}$. Since the energy-momentum tensors (2.3) are obtained by varying the diffeomorphism-invariant pieces of the action, each of them is independently conserved, hence one has on-shell $\nabla^\mu \mathcal{T}_{\mu\nu} = 0$, $\nabla^\mu T_{\mu\nu}^{(\Phi)} = 0$, $\nabla^\mu T_{\mu\nu}^{(\text{m})} = 0$.

Let us choose the FLRW ansatz for the metric,

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \quad (2.5)$$

with $K = 0, \pm 1$. Denoting $H = \dot{a}/a$ the Hubble parameter and $\psi = \dot{\Phi}$, the non-trivial gravitational equations (2.2) are

$$\begin{aligned} E_0^0 &\equiv -3M_{\text{Pl}}^2 \left(H^2 + \frac{K}{a^2} \right) + \frac{1}{2} \varepsilon \psi^2 - \frac{3}{2} \alpha \psi^2 \left(3H^2 + \frac{K}{a^2} \right) + \Lambda + \rho = 0, \\ E_1^1 &\equiv -M_{\text{Pl}}^2 \left(2\dot{H} + 3H^2 + \frac{K}{a^2} \right) - \frac{1}{2} \varepsilon \psi^2 \\ &\quad - \alpha \psi^2 \left(\dot{H} + \frac{3}{2} H^2 - \frac{K}{a^2} + 2H \frac{\dot{\psi}}{\psi} \right) + \Lambda - p = 0, \end{aligned} \quad (2.6)$$

and also $E_1^1 = E_2^2 = E_3^3$, while the scalar field equation (2.4) is

$$E_\Phi \equiv \frac{1}{a^3} \frac{d}{dt} \left(a^3 \left(3\alpha \left(H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi \right) = 0. \quad (2.7)$$

It is straightforward to check that

$$\dot{E}_0^0 + 3H(E_0^0 - E_1^1) + E_\Phi = 0, \quad (2.8)$$

in view of the matter conservation condition $\dot{\rho} + 3H(\rho + p) = 0$, hence only two of the three equations (2.6), (2.7) are independent. (These equations were recently applied to study a homogeneous collapse of the FRLW metric (2.5) with $K = +1$ matched to an exterior vacuum space-time to describe black hole formation from the point of view of an external observer [40].)

The first integral of the scalar field equation (2.7) is

$$a^3 \left(3\alpha \left(H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi = C, \quad (2.9)$$

where C is the Noether charge associated with the shift symmetry $\Phi \rightarrow \Phi + \Phi_0$. Let us first set $C = 0$. One finds in this case two different solutions which we shall call GR branch and screening branch. It turns out that solutions with $C \neq 0$ always approach one of these branches at late times.

The GR branch is obtained by setting $\psi = 0$. This solves the scalar equation (2.9), while the gravitational equation (2.6) reduces to

$$H^2 + \frac{K}{a^2} = \frac{\Lambda + \rho}{3M_{\text{Pl}}^2}. \quad (2.10)$$

Since the matter density ρ tends to zero at late time, the expansion is driven by Λ .

The screening branch is obtained by setting to zero the expression in the parenthesis in (2.9),

$$H^2 + \frac{K}{a^2} = \frac{\varepsilon}{3\alpha}. \quad (2.11)$$

This solves the scalar field equation, but the solution determines the metric and not the scalar field. The latter is determined by the gravitational equation (2.6),

$$\psi^2 = \frac{\alpha(\Lambda + \rho) - \varepsilon M_{\text{Pl}}^2}{\alpha(\varepsilon - 3\alpha K/a^2)}. \quad (2.12)$$

The role of the cosmological constant is now played by $\varepsilon/3\alpha$ while the Λ -term is screened and makes no contribution to the universe acceleration. Note that the matter density ρ is screened in the same sense, too. This applies for all (spatially open, closed, and flat) types of solutions, hence the spacetime is that of a constant curvature, de Sitter or anti-de Sitter, depending on the relative sign of ε and α (we shall see below that the absence of ghost requires that $\varepsilon \geq 0$ and $\alpha \geq 0$).

If $\varepsilon = 0$ then the F5 theory becomes F4 and the following *flat metric* configuration solves the field equations (which can be seen from Eqs.(2.11),(2.12)),

$$K = -1, \quad a = t, \quad \psi^2 = \frac{\Lambda + \rho}{3\alpha} t^2. \quad (2.13)$$

This is the principal virtue of the F4 theory – to admit a flat solution despite the non-zero Λ , hence the cosmological term can be *totally screened* (as well as ρ) [28, 29].

Let us now see what happens if $C \neq 0$. Eq.(2.9) then yields

$$\psi = \frac{C}{a^3 \left[3\alpha \left(H^2 + \frac{K}{a^2} \right) - \varepsilon \right]}, \quad (2.14)$$

injecting which to (2.6) gives

$$3M_{\text{Pl}}^2 \left(H^2 + \frac{K}{a^2} \right) = \frac{C^2 \left[\varepsilon - 3\alpha \left(3H^2 + \frac{K}{a^2} \right) \right]}{2a^6 \left[\varepsilon - 3\alpha \left(H^2 + \frac{K}{a^2} \right) \right]^2} + \Lambda + \rho. \quad (2.15)$$

This equation determines the algebraic dependence of the Hubble parameter H on the scale factor a . The relation to the physical time is then determined by the quadrature

$$t = \int \frac{da}{aH(a)}. \quad (2.16)$$

Let us set

$$H^2 = H_0^2 y, \quad a = a_0 a, \quad \rho_{\text{cr}} = 3M_{\text{Pl}}^2 H_0^2, \quad (2.17)$$

where H_0 and a_0 are the actual values of the Hubble parameter and of the scale factor, whereas ρ_{cr} is the critical density. Assuming the matter to be a mixture of a radiation and a non-relativistic component,

$$\rho = \rho_{\text{cr}} \left(\frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} \right), \quad (2.18)$$

Eq.(2.15) becomes

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6 \left[\zeta - 3y + \frac{\Omega_2}{a^2} \right]}{a^6 \left[\zeta - y + \frac{\Omega_2}{a^2} \right]^2}, \quad (2.19)$$

where

$$\Omega_0 = \frac{\Lambda}{\rho_{\text{cr}}}, \quad \Omega_2 = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_6 = \frac{C^2}{6\alpha a_0^6 H_0^2 \rho_{\text{cr}}}, \quad \zeta = \frac{\varepsilon}{3\alpha H_0^2}. \quad (2.20)$$

We assume in this text that $\Lambda > 0$, hence Ω_0 is always positive. The sign of Ω_2 is opposite to that of K , while the sign of Ω_6 is the same as that of α .

The scalar field ψ in (2.14) can be expressed in terms of a dimensionless Ψ as

$$\psi = -\sqrt{\frac{2\rho_{\text{cr}}\Omega_6}{3\alpha H_0^2}} \Psi \quad \text{where} \quad \Psi = \frac{1}{a[a^2(\zeta - y) + \Omega_2]}. \quad (2.21)$$

Before studying Eq.(2.19), let us introduce the conditions for the absence of ghosts derived in Appendix A. The ghosts may arise for generic perturbations of solutions of (2.19). However, they will be absent for the tensor and vector perturbations if $\alpha \geq 0$, hence if

$$\Omega_6 \geq 0, \quad (2.22)$$

while their absence in the scalar sector requires that

$$\mathcal{G} \equiv [y(y_* - y)^2 a^6 + \Omega_6(6y - y_*)][(y_* - y)^3 a^6 + \Omega_6(3y + y_*)] > 0, \quad (2.23)$$

where

$$y_* = \zeta + \frac{\Omega_2}{a^2}. \quad (2.24)$$

Therefore, our goal is to study solutions of the algebraic equation (2.19) subject to conditions (2.22) and (2.23). Bringing all terms in (2.19) to the common denominator yields

$$\frac{P(a, y)}{a^8 [y - y_*]^2} = 0, \quad (2.25)$$

where

$$P(a, y) = c_3(a) y^3 + c_2(a) y^2 + c_1(a) y + c_0(a) \quad (2.26)$$

is the cubic in y polynomial with the coefficients

$$\begin{aligned} c_3 &= -a^8, \quad c_2 = (\Omega_2 + 2\zeta) a^8 + 3\Omega_2 a^6 + \Omega_3 a^5 + \Omega_4 a^4, \quad c_1 = -\zeta(2\Omega_0 + \zeta) a^8 \\ &\quad - 2\Omega_2(\Omega_0 + 2\zeta) a^6 - 2\zeta\Omega_3 a^5 - (3\Omega_2^2 + 2\zeta\Omega_4) a^4 - 2\Omega_2\Omega_3 a^3 - (2\Omega_2\Omega_4 + 3\Omega_6) a^2, \\ c_0 &= \zeta^2\Omega_0 a^8 + \zeta\Omega_2(2\Omega_0 + \zeta) a^6 + \zeta^2\Omega_3 a^5 + (\Omega_4\zeta^2 + \Omega_2^2(\Omega_0 + 2\zeta)) a^4 + 2\zeta\Omega_2\Omega_3 a^3 \\ &\quad + (\Omega_2^3 + 2\zeta\Omega_2\Omega_4 + \zeta\Omega_6) a^2 + \Omega_2^2\Omega_3 a + \Omega_2(\Omega_2\Omega_4 + \Omega_6). \end{aligned} \quad (2.27)$$

Eq.(2.25) will be fulfilled if $P(a, y) = 0$ and $y \neq y_*$, hence the problem reduces to studying roots of the cubic polynomial. We notice that a cubic polynomial always has one real root, and it will have two more real roots if its coefficients fulfill the following two conditions,

$$\begin{aligned} \Delta &= (c_2)^2 - 3c_1c_3 > 0, \\ D &= 27(c_0c_3)^2 - 18c_0c_1c_2c_3 + 4c_0(c_2)^3 + 4(c_1)^3c_3 - (c_1c_2)^2 < 0. \end{aligned} \quad (2.28)$$

These conditions insure that the polynomial has two real extrema of the opposite sign.

3 Constructing the solutions

Our next task is to solve Eq.(2.19) (see [41] for the $\Omega_2 = \Omega_4 = 0$ case), for which we shall first analyse the limits where the scale factor a is either large or small, and then construct the solutions globally.

3.1 Late time limit $a \rightarrow \infty$

If a is large then there is always a solution of (2.19) approaching the GR branch (2.10),

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{(\zeta - 3\Omega_0)\Omega_6}{(\Omega_0 - \zeta)^2 a^6} + \mathcal{O}\left(\frac{1}{a^7}\right), \quad (3.1)$$

the corresponding dimensionless scalar being

$$\Psi = \frac{1}{(\zeta - \Omega_0)a^3} + \mathcal{O}\left(\frac{1}{a^6}\right). \quad (3.2)$$

The ghost function \mathcal{G} (2.23) reduces in the leading order to

$$\mathcal{G} = \Omega_0(\zeta - \Omega_0)^5 a^{12} + \dots > 0, \quad (3.3)$$

hence, since $\Omega_0 > 0$, ghost will be absent if $\zeta > \Omega_0$.

Next, Eq.(2.28) yields in the leading order

$$\Delta = (\Omega_0 - \zeta)^2 a^{16} + \dots, \quad D = -8\Omega_6\zeta(\Omega_0 - \zeta)^3 a^{26} + \dots,$$

where the dots denote subleading terms. Hence, as $\Omega_6 > 0$, conditions (2.28) are fulfilled if

$$\zeta(\Omega_0 - \zeta) > 0, \quad (3.4)$$

in which case there are two more solutions $h(a)$ at large a . They approach the screening branch (2.11) and have the structure

$$y_{\pm} = \zeta + \frac{\Omega_2}{a^2} \pm \frac{\chi}{(\Omega_0 - \zeta)a^3} \pm \frac{\Omega_2\Omega_6}{\chi a^5} - \frac{\Omega_6(\zeta - 3\Omega_0) \pm \Omega_3\chi}{2(\Omega_0 - \zeta)^2 a^6} + \mathcal{O}\left(\frac{1}{a^7}\right), \quad (3.5)$$

with $\chi = \sqrt{2\zeta\Omega_6(\Omega_0 - \zeta)}$, while

$$\Psi_{\pm} = \mp \sqrt{\frac{\Omega_0 - \zeta}{2\zeta\Omega_6}} + \mathcal{O}\left(\frac{1}{a^2}\right). \quad (3.6)$$

Calculating again the leading terms of the function \mathcal{G} in (2.23) shows that both y_{\pm} will be ghost-free if

$$5 + \frac{2\zeta}{\Omega_0 - \zeta} > 0. \quad (3.7)$$

These facts can be summarized by dividing the values of ζ into three regions as follows.

I: $\zeta > \Omega_0 \Rightarrow$ only the GR solution (3.1) exists – stable (no ghost).

II: $\zeta < 0 \Rightarrow$ only the GR solution (3.1) exists – unstable.

III: $0 < \zeta < \Omega_0 \Rightarrow$ there are three solutions. The GR solution (3.1) is unstable but the two screening solutions (3.5) are stable.

One should emphasize that by “stable solutions” we mean, in this Section only, solutions which do not show the ghost instability. However, they may have other instabilities, which will be discussed in the next Section.

Let us now consider the special cases $\zeta = 0$ and $\zeta = \Omega_0$. If $\zeta = \Omega_0 \neq 0$ then there is only one solution,

$$y = \zeta + \frac{\Omega_2 + \xi}{a^2} + \frac{\Omega_3}{3a^3} + \mathcal{O}\left(\frac{1}{a^4}\right), \quad \Psi = -\frac{1}{\xi a} + \mathcal{O}\left(\frac{1}{a^2}\right), \quad (3.8)$$

where $\xi = (-2\zeta\Omega_3)^{1/3}$. This solution is stable.

For $\zeta = 0$ there are three different solutions. Let us first assume that $\Omega_2 \neq 0$. Then one solution is obtained by simply setting $\zeta = 0$ in Eqs.(3.1),(3.2), and Eq.(3.3) then shows that this solution is unstable. The two other solutions are

$$y_{\pm} = \frac{\Omega_2}{a^2} \pm \frac{\xi}{a^4} + \frac{3\Omega_6}{2\Omega_0 a^6} \mp \frac{\xi\Omega_3}{2\Omega_0 a^7} + \mathcal{O}\left(\frac{1}{a^8}\right), \quad \Psi_{\pm} = \mp \frac{a}{\xi} + \mathcal{O}\left(\frac{1}{a}\right), \quad (3.9)$$

where $\xi = \sqrt{2\Omega_2\Omega_6/\Omega_0}$. These solutions are stable, since $\mathcal{G} = 20\Omega_2^2\Omega_6^2/a^4 + \dots$. Both of them approach flat geometry if $\Omega_2 > 0$ ($K = -1$).

If $\Omega_2 = \zeta = 0$ then solutions can be obtained analytically, these are $y = 0$ and

$$y_{\pm} = \frac{1}{2} \left(\Omega_0 + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} \pm \sqrt{\left(\Omega_0 + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} \right)^2 - \frac{12\Omega_6}{a^6}} \right). \quad (3.10)$$

In the latter case Eq.(2.23) gives $\mathcal{G} = y^2(y^2a^6 + 6\Omega_6)(3\Omega_6 - y^2a^6)$. The $y = 0$ configuration is actually an artefact and not a solution of the original problem because ψ in (2.21) is not defined if $\zeta = y = \Omega_2 = 0$. Next, one has $y_+ \rightarrow \Omega_0$ and hence $\mathcal{G} \rightarrow -\Omega_0^6 a^{12}$ at large a , therefore this solution is unstable. The y_- solution has the following behaviour at large and small a ,

$$\frac{3\Omega_6}{(\Omega_0 a^4 + \Omega_3 a + \Omega_4) a^2} \leftarrow y_- \rightarrow \frac{3\Omega_6}{\Omega_0 a^6}, \quad (3.11)$$

hence \mathcal{G} is positive at large a but it can be negative at small a , for example if $\Omega_3 = \Omega_4 = 0$.

The above analysis exhausts all possible types of the late time solutions. However, for $\zeta = 0$ there is one more solution – flat space described by (2.13). This solution should be considered separately, because it has $C = \Omega_6 = 0$ but $\psi \neq 0$. Inserting this solution to Eq.(A.21) in the Appendix A gives the positive eigenvalue of the kinetic energy matrix,

$$\lambda_1 = t(2M_{\text{Pl}}^2 + 5\alpha\psi^2) > 0, \quad (3.12)$$

which implies that Minkowski space is ghost-free. However, it may be unstable within the full F4 theory – the corresponding no-ghost conditions are given in Appendix B.

Summarizing, stable late time solutions for generic values of α are either

$$y = \Omega_0 + \dots, \quad \Psi = \frac{1}{(\alpha - \Omega_0) a^3} + \dots \quad (3.13)$$

if $\alpha > \Omega_0$ or

$$y = \zeta + \dots, \quad \Psi = \pm \sqrt{\frac{\Omega_0 - \zeta}{2\zeta \Omega_6}} + \dots \quad (3.14)$$

if $0 < \alpha < \Omega_0$.

3.2 Limit $a \rightarrow 0$

Let us now consider the $a \rightarrow 0$ limit, assuming first that $\Omega_4 \neq 0$. Then there is always the GR type solution,

$$y = \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} + \frac{\Omega_2 \Omega_4 - 3\Omega_6}{\Omega_4 a^2} + \frac{3\Omega_3 \Omega_6}{\Omega_4 a} + \mathcal{O}(1), \quad \Psi = -\frac{a}{\Omega_4} + \mathcal{O}(a^2), \quad (3.15)$$

but it is unstable since $\mathcal{G} = -\Omega_4^6/a^{12} + \dots$. Next, computing the leading terms of the Δ, D coefficients (2.28) one finds that there should be two more solutions if

$$8\Omega_2 \Omega_4 + 9\Omega_6 > 0. \quad (3.16)$$

If this condition is fulfilled, then introducing $\sigma = \sqrt{\Omega_6(8\Omega_2 \Omega_4 + 9\Omega_6)}$, the two other solutions read

$$y_{\pm} = \frac{2\Omega_2 \Omega_4 + 3\Omega_6 \pm \sigma}{2\Omega_4 a^2} \mp \frac{\Omega_3 \Omega_6 (4\Omega_2 \Omega_4 + 9\Omega_6 \pm 3\sigma)}{2\sigma \Omega_4^2 a} + \mathcal{O}(1), \quad \Psi_{\pm} = -\frac{2\Omega_4}{(3\Omega_6 \pm \sigma)a} + \mathcal{O}(1), \quad (3.17)$$

whose stability conditions are, respectively,

$$(5\Omega_2 \Omega_4 \pm 3\sigma + 9\Omega_6)(8\Omega_2 \Omega_4 \pm 3\sigma + 9\Omega_6) > 0. \quad (3.18)$$

These solutions can be called screening, since the matter contribution, usually dominant at small a , is screened. If $\Omega_2 = 0$ then nothing special happens to the GR solution (3.15) and it remains unstable, while the screening solutions y_{\pm} become

$$\begin{aligned} y_+ &= \frac{3\Omega_6}{\Omega_4 a^2} - \frac{3\Omega_3 \Omega_6}{\Omega_4^2 a} + \frac{5}{3} \zeta + \frac{3\Omega_6 \Omega_3^2 + 9\Omega_6^2}{\Omega_4^3} + \mathcal{O}(a), & \Psi_+ &= -\frac{\Omega_4}{3\Omega_6 a} + \mathcal{O}(1), \\ y_- &= \frac{\zeta}{3} + \frac{4\zeta^2}{27\Omega_6} (\Omega_4 a^2 + \Omega_3 a^3) + \mathcal{O}(a^4), & \Psi_- &= \frac{3}{2\zeta a^3} + \mathcal{O}\left(\frac{1}{a}\right), \end{aligned} \quad (3.19)$$

and these are both stable.

Let us now set $\Omega_4 = 0$. Then there is always a screening solution,

$$y = \frac{\Omega_2}{3a^2} + \frac{4\Omega_2^2 \Omega_3}{27\Omega_6 a} + \frac{\zeta}{3} + \frac{8\Omega_2^3}{81\Omega_6} - \frac{16\Omega_2^3 \Omega_3^2}{243\Omega_6^2} + \mathcal{O}(a), \quad \Psi = \frac{3}{2\Omega_2 a} + \mathcal{O}(1), \quad (3.20)$$

with $\mathcal{G} = 2\Omega_2\Omega_6^2/a^4 + \dots$, hence it is stable. Computing again the leading terms of Δ, D in (2.28), one finds two more solutions if

$$\Omega_3^2 - 12\Omega_6 > 0. \quad (3.21)$$

Introducing $\omega = \sqrt{\Omega_3^2 - 12\Omega_6}$ these solutions are

$$y_{\pm} = \frac{\Omega_3 \pm \omega}{2a^3} + \frac{\Omega_2(\Omega_3^2 \pm \omega\Omega_3 - 16\Omega_6)}{\omega(\omega \pm \Omega_3)a^2} + \mathcal{O}\left(\frac{1}{a}\right), \quad \Psi_{\pm} = -\frac{2}{\Omega_3 \pm \omega} + \mathcal{O}(a), \quad (3.22)$$

with the stability condition

$$(\Omega_3^2 \pm \omega\Omega_3 + 6\Omega_6)(\Omega_3^2 \pm \omega\Omega_3 - 12\Omega_6) < 0. \quad (3.23)$$

A simple analysis of this condition shows that y_- is always stable while y_+ is always unstable. If $\Omega_2 = 0$ then (3.20), (3.22) reduce to

$$\begin{aligned} y &= \frac{\zeta}{3} + \frac{4\zeta^2\Omega_3}{27\Omega_6}a^3 + \mathcal{O}(a^6), & \Psi &= \frac{3}{2\eta a^3} + \mathcal{O}(1), \\ y_{\pm} &= \frac{\Omega_3 \pm \omega}{2a^3} + \Omega_0 + \Omega_6 \frac{6\Omega_0 - 10\zeta}{\omega(\omega \pm \Omega_3)} + \mathcal{O}(a^3), & \Psi_{\pm} &= \frac{2}{\Omega_3 \pm \omega} + \mathcal{O}(a^3), \end{aligned} \quad (3.24)$$

where y, y_- are stable and y_+ is unstable.

Let us finally assume that $\underline{\Omega_3 = \Omega_4 = 0}$. Then there is only one solution,

$$y = \frac{\Omega_2}{3a^2} + \frac{\zeta}{3} + \frac{8\Omega_2^3}{81\Omega_6} + \left(\frac{4\Omega_2^2(\Omega_0 + \zeta)}{27\Omega_6} - \frac{32\Omega_2^5}{729\Omega_6^2} \right) a^2 + \mathcal{O}(a^4), \quad \Psi = \frac{3}{2\Omega_2 a} + \mathcal{O}(a), \quad (3.25)$$

and it is stable. For $\Omega_2 = 0$ it remains stable and reduces to

$$y = \frac{\zeta}{3} + \frac{4\zeta^2(3\Omega_0 - \zeta)}{81\Omega_6}a^6 + \mathcal{O}(a^{12}), \quad \Psi = \frac{3}{2\zeta a^3} + \mathcal{O}(a^3). \quad (3.26)$$

Let us summarize the above results in the case where $K = \Omega_2 = 0$. Assuming generic values of the other parameters, the stable near the singularity solution is

$$y = \frac{\zeta}{3} + \dots, \quad \Psi = \frac{3}{2\zeta a^3} + \dots, \quad (3.27)$$

which is of the inflationary type. If $\Omega_4 \neq 0$ then there is one more ghost-free solution

$$y = \frac{3\Omega_6}{\Omega_4 a^2} + \dots, \quad \Psi = -\frac{\Omega_4}{3\Omega_6 a} + \dots, \quad (3.28)$$

hence $a(t) \sim t$; and if $\Omega_4 = 0$ but $\omega^2 = \Omega_3^2 - 12\Omega_6 > 0$ this solution reduces to

$$y = \frac{\Omega_3 - \omega}{2a^3} + \dots, \quad \Psi = \frac{2}{\Omega_3 - \omega} + \dots, \quad (3.29)$$

hence $a(t) \sim t^{2/3}$. We note that the standard radiation-dominated solution (3.15) has ghost and is eliminated from consideration, while the ghost-free solutions (3.27)–(3.29) do not show a radiation-dominated phase with $y \sim 1/a^4$. Without this phase the model cannot correctly describe the primary nucleosynthesis.

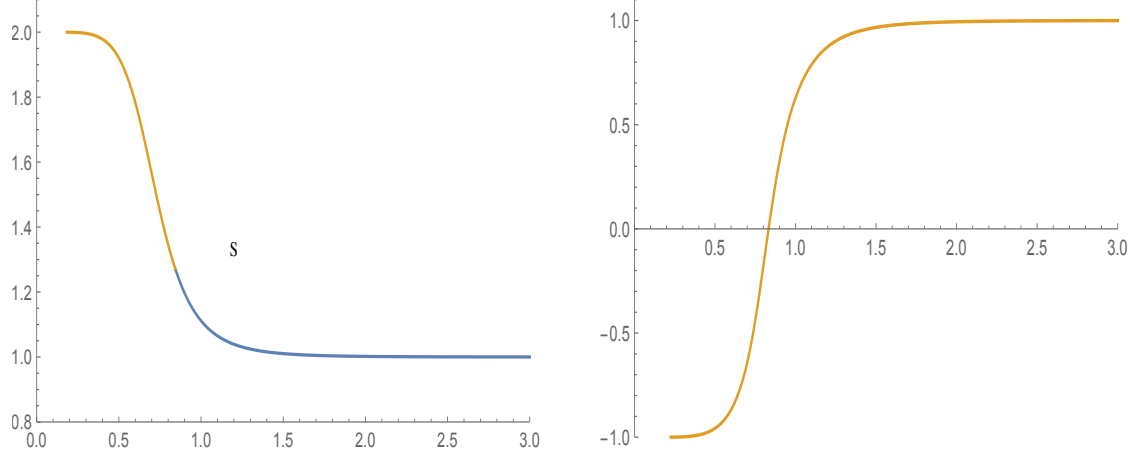


Figure 1. Solutions $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = 0$, $\Omega_3 = \Omega_4 = 0$ and for $\zeta = 6$ (left panel) or $\zeta = -3$ (right panel).

3.3 Global solutions

We now know that the ghost-free solutions are described by Eqs.(3.27)–(3.29) near the singularity and by Eqs.(3.13),(3.14) at late times. Having understood their asymptotic structure, we can construct the solutions globally, for example by using the Cardano formula. We shall start by describing all possible solution types and later select those which are ghost-free.

Let us assume at first that there is no matter, $\Omega_3 = \Omega_4 = 0$. Then there is only one solution (3.26) near the singularity. One can always rescale the system to set $\Omega_0 = 1$. It turns out that the qualitative behaviour of the solutions is insensitive to the value of Ω_6 , as long as $\Omega_6 > 0$, hence one can set $\Omega_6 = 1$. At the same time, the value of ζ is important as it determines the number of solutions at infinity. If $\zeta > \Omega_0 = 1$ or $\zeta < 0$ then there is just one solution at infinity, and hence only one global solution. If $\zeta > \Omega_0$ then this is the solution of the type S shown in Fig.1. It has asymptotics

$$\frac{\zeta}{3} \leftarrow y \rightarrow \Omega_0, \quad (3.30)$$

hence the universe inflates with constant but different Hubble rates at early times and at late times. The function \mathcal{G} is everywhere positive, therefore this solution is stable.

If $\zeta < 0$ then the solution is described by the curve of type a shown in the right panel of Fig.1. Such a solution makes sense only in the region $a > a_{\min}$ where $y = (H/H_0)^2 > 0$. Since $H = \pm H_0 \sqrt{y}$ can have both signs in this region, the solution describes a bounce – a universe contracting with $H = -H_0 \sqrt{y}$ up to a minimal size a_{\min} and then expanding with $H = +H_0 \sqrt{y}$. (This behaviour is similar to that found in the scalar-tensor gravity with a negative scalar field potential [42].) However, such bounce solutions are unstable, since ghost is present for $\zeta < 0$.

Let us now choose a value $\zeta \in (0, \Omega_0)$. Then there are three local solutions at large a and only one at small a , hence there is only one solution that continues from the large a all the way down to small a . This corresponds to the solution A in Fig.2;

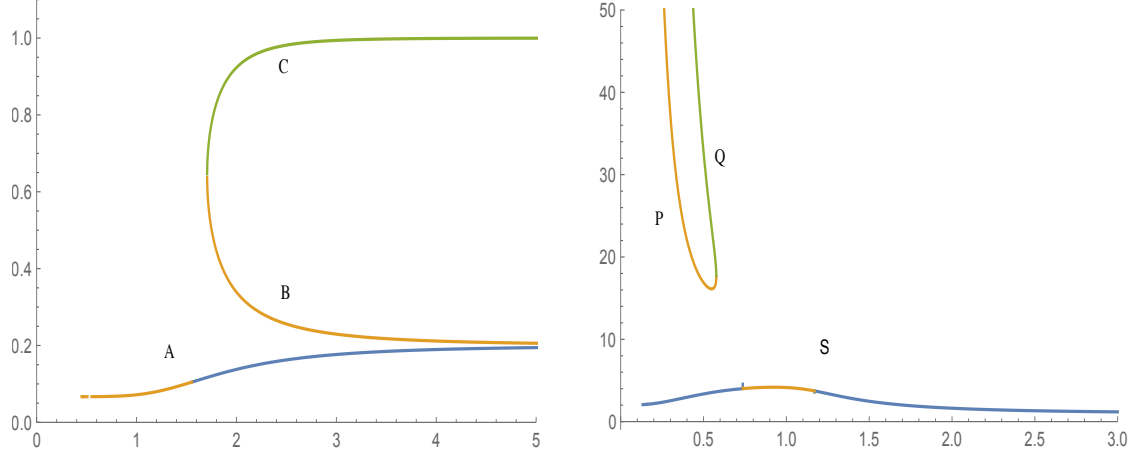


Figure 2. Solutions $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = 0$, $\Omega_3 = \Omega_4 = 0$, $\zeta = 0.2$ (left) and for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = 0$, $\Omega_3 = 5$, $\Omega_4 = 0$, $\zeta = 6$ (right).

it has asymptotics

$$\frac{\zeta}{3} \leftarrow y \rightarrow \zeta, \quad (3.31)$$

and it is stable. The two other solutions, B and C in Fig.2, exist only near infinity and cannot extend down to the singularity, hence they merge each other at some finite value $a = a_*$ where $y(a_*) = y_*$. One has near this point $a - a_* \sim (y - y_*)^2$ and hence

$$a(t) = a_* + a_* \sqrt{y_*} |t - t_*| + \mathcal{O}((t - t_*)^{3/2}). \quad (3.32)$$

The geometry is singular at the moment $t = t_*$ when the time “emerges” and it is not possible to regularly continue the spacetime to the $a < a_*$ region. (A similar situation occurs in the $F(R)$ gravity if $F''(R) = 0$ at some $R = R_*$ [10].) Such solutions are unlikely to be physically interesting, in addition the solution C is unstable. Summarizing, only the solutions S in Fig.1 and A in Fig.2 are stable.

Let us now add the matter by setting $\Omega_4 \neq 0$ and/or $\Omega_3 \neq 0$. This does not affect much solutions at large a , but this creates new solutions at small a . Let us again consider the $\zeta > \Omega_0$ case. Then there is only one solution at infinity but there are three of them near the singularity. Therefore, only one solution can extend to the whole interval of a , this is the solution S shown in the right panel of Fig.2. This solution is similar to the S in Fig.1, since it is also stable and has the same boundary conditions (3.30), while taking the matter into account only produces some deformations of the solution in the intermediate region. At the same time, the matter gives rise to two more solution near the singularity – solutions P and Q in the right panel of Fig.2 – they cannot extend to large a and hence merge each other at some point (“the end of time”). It is again unlikely that such solutions could be physically interesting, because one of them is unstable. Therefore, including the matter does not bring anything new at this point.

However, the matter can be essential if $\zeta \in (0, \Omega_0)$, since in this case the number of solutions at small a matches that at large a , hence there can be three different solutions

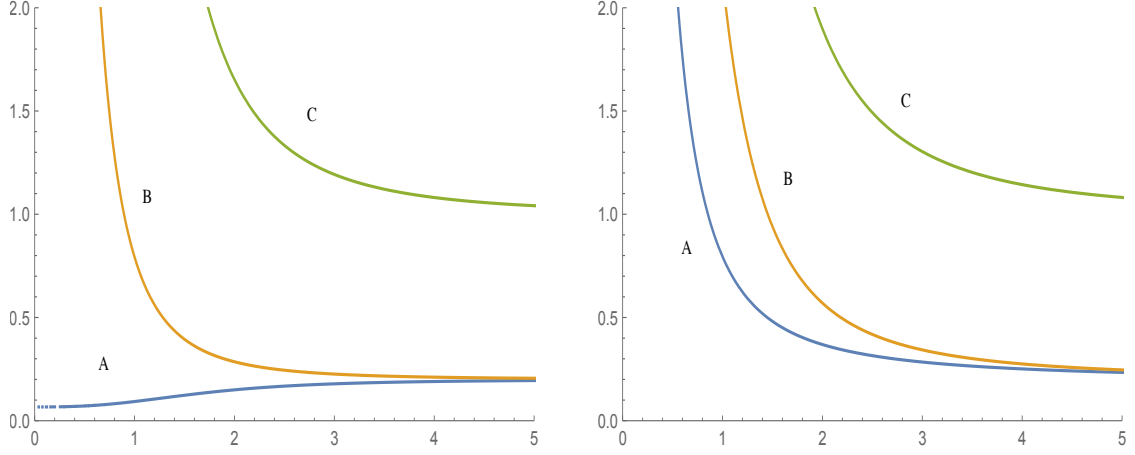


Figure 3. Solutions $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_3 = 5$, $\Omega_4 = 0$, $\zeta = 0.2$. One has $\Omega_2 = 0$ (left) and $\Omega_2 = 1$ (right).

extending to the whole interval of a . This is shown in Fig.3, where the solution A is essentially the same as before, in Fig.2. However, the solutions B and C no longer merge each other but extend all the way down to the singularity at $a = 0$ where they meet the corresponding local solutions. The solution C is unstable but the B is stable. Therefore, in addition to the solutions of types S and A with the asymptotics (3.30) and (3.31), there is a third stable global solution with the following behaviour,

$$\frac{3\Omega_6}{\Omega_4 a^2} \leftarrow y \rightarrow \zeta \quad \text{or} \quad \frac{\Omega_3 - \sqrt{\Omega_3^2 - 12\Omega_6}}{2a^3} \leftarrow y \rightarrow \zeta \quad \text{if } \Omega_4 = 0. \quad (3.33)$$

This exhaust all ghost-free solutions. All the above arguments applies also for $\Omega_2 \neq 0$, for example the three global solutions for $\Omega_2 > 0$ are shown in the right panel of Fig.3.

Let us finally note that, since one has at present $y = a = 1$, one should have

$$1 = \Omega_0 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_6 \frac{\zeta - 3 + \Omega_2}{[\zeta - 1 + \Omega_2]^2}. \quad (3.34)$$

However, since there can be several solutions $y(a)$, one cannot impose this condition beforehand. Indeed, the condition requires the $y(a)$ curve to pass through the physical point $(1, 1)$, but one does not know in advance which of the several curves is physical. Therefore, one has to first choose a particular solution $y(a)$ obtained for some parameter values Ω_k, ζ , then choose a point (a_*, y_*) on this curve, and then declare this point to be physical. Since one has

$$y_* = \Omega_0 + \frac{\Omega_2}{a_*^2} + \frac{\Omega_3}{a_*^3} + \frac{\Omega_4}{a_*^4} + \frac{\Omega_6 \left[\zeta - 3y_* + \frac{\Omega_2}{a_*^2} \right]}{a_*^6 \left[\zeta - y_* + \frac{\Omega_2}{a_*^2} \right]^2}, \quad (3.35)$$

this can be rewritten as the normalization condition

$$1 = \Omega_0^* + \Omega_2^* + \Omega_3^* + \Omega_4^* + \Omega_6^* \frac{\zeta^* - 3 + \Omega_2^*}{[\zeta^* - 1 + \Omega_2^*]^2}, \quad (3.36)$$

with the modified parameter values

$$\Omega_0^* = \frac{\Omega_0}{y_*}, \quad \Omega_2^* = \frac{\Omega_2}{y_* a_*^2}, \quad \Omega_3^* = \frac{\Omega_3}{y_* a_*^3}, \quad \Omega_4^* = \frac{\Omega_4}{y_* a_*^4}, \quad \Omega_6^* = \frac{\Omega_6}{y_*^2 a_*^4}, \quad \zeta^* = \frac{\zeta}{y_*}. \quad (3.37)$$

As a result, imposing the normalization condition is achieved by rescaling the parameters.

4 Stability of the solutions

Summarizing what was said above, the F5 theory (1.6) admits various cosmological solutions, but ghost-free solutions exist only if $\alpha \geq 0$ and $\varepsilon \geq 0$. The no-ghost conditions eliminate many solutions, as for example the bounces or the “emerging time” solutions. Setting $\varepsilon = 0$ gives a subset of the F4 theory where flat space is a ghost-free solution, but it may develop ghost within the full F4, as shown in Appendix B. Expanding cosmologies are obtained for $\varepsilon > 0$, and some of them still show ghost, but there are three different types of ghost-free solutions corresponding to the curves S, A, B in Figs. (1)–(3).

The S solution exists for $\zeta > \Omega_0$ and is sourced by the scalar field, but it may also contain the matter. It describes a universe with the standard late time dynamic dominated by the Λ -term, radiation and dust. At early times the matter effects are totally screened and the universe expands with a constant Hubble rate determined by ε/α . Since it contains two independent parameters ζ and $\Omega_0 \sim \Lambda$ in the asymptotics, this solution can have an hierarchy between the Hubble scales at the early and late times. However, at late times it is not screening and dominated by Λ , thus invoking again the cosmological constant problem.

The solutions A and B exist for $0 < \zeta < \Omega_0$. The solution A is sourced by the scalar field, with or without the matter, while the solution B exists only when the matter is present. They both show the screening because their late time behaviour is controlled by $\zeta \sim \varepsilon/\alpha$ and not by Λ . Therefore, they could in principle describe the late time acceleration while circumventing the cosmological constant problem, and one might probably find arguments justifying that ε/α should be small. At the same time, these solutions cannot describe the early inflationary phase. Indeed, the near singularity behaviour (3.33) of the solution B does not correspond to inflation, while the solution A does show an inflationary phase, but with essentially the same Hubble rate as at late times, hence there is no hierarchy between the two Hubble scales. In addition, as was mentioned above, none of the ghost-free solutions has a radiation-dominated phase needed for the nucleosynthesis. As we shall now see, there are other problems near the singularity.

The solutions S, A, B do not have the ghost instability. However, this does not mean they are stable, since they may have other instabilities. Let us therefore study their stability. The linear equations for generic perturbations of spatially flat homogeneous and isotropic backgrounds are derived in Appendix C (the $K = 0$ case is the most important). These equations describe the scalar and tensor perturbations, but we shall illustrate the procedure by discussing only the tensor sector, since the

equations are simpler in this case. Each of the two graviton polarizations is described by the same equation (C.11),

$$(\Omega_6 \Psi^2 + 1) \ddot{w} + \left(2\Omega_6 \Psi \dot{\Psi} + 3(\Omega_6 \Psi^2 + 1)h \right) \dot{w} - \left(2(\Omega_6 \Psi^2 + 1)(2\dot{h} + 3h^2) + 2\Omega_6(3\zeta \Psi^2 + 4h\Psi\dot{\Psi}) - 6\Omega_0 + \frac{P^2}{a^2}(\Omega_6 \Psi^2 - 1) \right) w = 0, \quad (4.1)$$

where the derivatives are taken with respect to the dimensionless time $\tau = H_0 t$. The equations in the scalar sector (Eqs.(C.8)–(C.10)) are more complicated but their solutions are qualitatively similar.

The coefficients in (4.1) depend on the Hubble parameter h and scalar Φ , which are determined by the background solutions. The solutions will be stable if their perturbations are bounded. Now, since their equations are linear, the perturbations can become unbounded only asymptotically, either in the past, when $a \rightarrow 0$, or in the future, when $a \rightarrow \infty$. It is therefore sufficient to study the perturbations only in the small time limit when the background solutions are described by Eqs.(3.27)–(3.29), and also in the late time limit when the backgrounds are described by (3.13) or by (3.14).

Let us first check if solutions of (4.1) are bounded for $a \rightarrow \infty$. At late times one can neglect the P^2/a^2 term, while h, Ψ are then given either by (3.13) or by (3.14). Consider first the GR branch (3.13). One has in this case $\Psi \sim 1/a^3$, hence for $a \rightarrow \infty$ one can set $\Psi = 0$, while $h = \sqrt{\Omega_0}$. As a result, Eq.(4.1) reduces to $\ddot{w} + 3h\dot{w} = 0$, whose solution

$$w = C_1 + C_2 e^{-3h\tau} \quad (4.2)$$

is bounded as $\tau \rightarrow \infty$. Therefore, all tensor modes are bounded at late times. Considering similarly the scalar perturbation modes w, u, ϕ described by Eqs.(C.8)–(C.10) one finds

$$w = C_1, \quad u = C_2, \quad \phi = C_3 + C_4 e^{-3h\tau}, \quad (4.3)$$

and hence w, u and $\dot{\phi}/\Psi$ are bounded, too. The conclusion is that the GR branch (3.13) is dynamically stable.

For the screening branch (3.14) one has

$$h = \sqrt{\zeta}, \quad \Psi = \pm \sqrt{\frac{\Omega_0 - \zeta}{2\zeta\Omega_6}}, \quad (4.4)$$

injecting which to Eq.(4.1) reduces the equation again to $\ddot{w} + 3h\dot{w} = 0$, hence the solution is again $w = C_1 + C_2 e^{-3h\tau}$. Considering the scalar modes w, u, ϕ , one finds that they are bounded for $\tau \rightarrow \infty$ as well. Therefore, the screening branch is stable, too.

Let us now see if the solutions are stable in the past, when $a \rightarrow 0$. The corresponding background solutions can be of three different types described by (3.27)–(3.29). For small a one cannot neglect the P^2/a^2 term anymore, but let us first analyse the homogeneous modes with $P = 0$. For the screening branch described by (3.27) one

has

$$h = \sqrt{\zeta/3}, \quad \Psi = \frac{3}{\zeta a^3}, \quad (4.5)$$

inserting which to Eq.(4.1) gives

$$\frac{\Omega_6}{h^4 a^6} (\ddot{w} - 3\dot{w}) + \ddot{w} + 3h\dot{w} + 6(\Omega_0 - h^2)w = 0. \quad (4.6)$$

When a is small, the term proportional to $1/a^6$ is dominant and hence the equation reduces to $\ddot{w} - 3h\dot{w} = 0$ whose solution $w = C_1 + C_2 e^{+3h\tau}$ is bounded for $\tau \rightarrow -\infty$. The scalar sector amplitudes w, u, ϕ are bounded as well. Therefore, this branch is stable with respect to homogeneous perturbations.

For the second branch described by (3.28) one has

$$a(\tau) = \sqrt{\frac{3\Omega_6}{\Omega_4}} \tau, \quad \Psi = -\frac{\Omega_4}{3\Omega_6 a}, \quad (4.7)$$

inserting which to Eq.(4.1) and keeping only the leading at small a terms gives

$$\ddot{w} + \frac{1}{t} \dot{w} + \frac{6}{t^2} w = 0. \quad (4.8)$$

The solution is $w = C_1 \cos(\sqrt{6} \ln(\tau) + C_2)$ and this is bounded as $\tau \rightarrow 0$, however its derivatives grow without bounds and hence the curvature blows up. In addition, solving for the scalar modes w, u, ϕ one finds that ϕ contains a piece proportional to $1/\tau^3$, hence $\dot{\phi}/\Psi$ is unbounded. Since the perturbations grow, this solution branch is unstable.

For the third branch (3.29) one has $a(\tau) \sim \tau^{2/3}$ and $\Psi = \mathcal{O}(1)$, in which case Eq.(4.1) yields $w \sim 1/\tau$, hence this branch is unstable, too.

Summarizing, so far only the screening branch (3.27) passes the stability check. This branch is actually the most interesting, however, its stability is not yet established, since there remains to consider the inhomogeneous perturbations with $P \neq 0$. Let us therefore return to Eq.(4.5) and insert it to Eq.(4.1). The result is

$$\frac{\Omega_6}{h^4 a^6} \left(\ddot{w} - 3\dot{w} - \frac{P^2}{a^2} w \right) + \ddot{w} + 3h\dot{w} + 6(\Omega_0 - h^2)w + \frac{P^2}{a^2} w = 0, \quad (4.9)$$

and it is clear that the terms proportional to Ω_6 are dominant when $a \rightarrow 0$, hence the equation reduces to

$$\ddot{w} - 3\dot{w} - \frac{P^2}{a^2(\tau)} w = 0 \quad \text{with} \quad a(\tau) = e^{h\tau}. \quad (4.10)$$

The solution of this equation

$$w = C_1 a^2(\tau) [P - ha(\tau)] \exp\left(\frac{P}{ha(\tau)}\right) + C_2 a^2(\tau) [P + ha(\tau)] \exp\left(-\frac{P}{ha(\tau)}\right) \quad (4.11)$$

diverges as $a(\tau) \rightarrow 0$, and the divergence is very strong – it is proportional to the exponent of exponent of τ . This effect is produced by terms proportional to Ω_6 , hence by the background scalar. Therefore, the screening solution (4.5) is unstable as well.

As a result, we conclude that all isotropic ghost-free solutions in the theory are unstable in the vicinity of the initial spacetime singularity. We know that in GR the isotropic cosmologies are also unstable near the singularity (hence one should study more general anisotropic solutions), but in our case the instability is stronger – it is exponential and not power law as in GR. Therefore, the F5 theory does not have viable isotropic solutions describing the whole of the cosmological history. This conclusion is supported by the previous observation that the model does not have a radiation-dominated phase and hence cannot correctly describe the primary nucleosynthesis. On the other hand, one cannot perhaps expect a particular field theory model to be able to describe everything, whereas at late times it admits stable solutions with an accelerating phase. For $0 < \zeta < \Omega_0$ these solutions show the screening, since their Hubble parameter is determined not by the conventional Λ -term but by $\zeta \sim \varepsilon/\alpha$, which circumvents the cosmological constant problem. Hence the model fulfills the necessary for it condition – to provide an explanation for the current cosmic acceleration.

Therefore, the main outcome of our analysis is the conclusion that the Horndeski theory may indeed offer interesting for cosmology features. Although the model we considered is not totally satisfactory, one may hope that more realistic models can be obtained by adjusting the coefficient functions $G_k(X, \Phi)$ in the Horndeski Lagrangian (1.2). Unfortunately, to decide whether or not a given model is realistic always requires to carry out a tedious analysis, similar to the one presented above.

Acknowledgments

We thank Gary Gibbons for discussions and a reading of the manuscript. MSV was partly supported by the Russian Government Program of Competitive Growth of the Kazan Federal University. AAS and SVS were supported by the RSF grant 16-12-10401. SVS kindly appreciates the hospitality of the LMPT at the University of Tours.

A The no-ghost conditions

These conditions guarantee that the kinetic energy is positive. At the perturbative level, this means that the kinetic part of the second variation of the Lagrangian should be described by a positive-definite quadratic form. The second variation of the Lagrangian can be obtained by perturbing the background configuration,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \quad \Phi \rightarrow \Phi + \delta\Phi, \quad (\text{A.1})$$

computing the linearized field equations for the perturbations $\delta g_{\mu\nu}$ and $\delta\Phi$,

$$\begin{aligned} \delta E_{\mu\nu} &\equiv \delta(M_{\text{Pl}}^2 G_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha \mathcal{T}_{\mu\nu} - \varepsilon T_{\mu\nu}^{(\Phi)} - T_{\mu\nu}^{(\text{m})}), \\ \delta E_\Phi &\equiv \delta(\nabla_\mu((\alpha G^{\mu\nu} + \varepsilon g^{\mu\nu}) \nabla_\nu \Phi)), \end{aligned} \quad (\text{A.2})$$

and using this to compute

$$\delta^2 S = \frac{1}{2} \int (\delta E_{\mu\nu} \delta g^{\mu\nu} + \delta E_\Phi \delta \Phi) \sqrt{-g} d^4 x \equiv \frac{1}{2} \int \delta^2 L d^4 x. \quad (\text{A.3})$$

The resulting $\delta^2 L$ splits into three independent parts corresponding to contributions of the scalar, vector, and tensor modes, and the positivity of the kinetic terms imposes an independent condition in each sector.

Before considering generic perturbations, it is instructive to see that there is a simple way to obtain the correct answer by considering only anisotropic perturbations. Let us assume the spacetime metric to be

$$ds^2 = -N^2 dt^2 + a_1^2 dx^2 + a_2^2 dx_2^2 + a_3^2 dx_3^2, \quad (\text{A.4})$$

where N, a_k are functions of t , while $\Phi = \Phi(t)$. This can be viewed as an anisotropic deformation of the isotropic metric with $K = 0$, and for the generic a_k such a deformation contains contributions from both scalar and tensor perturbation sectors. Dropping a total derivative and neglecting the matter contribution, the Lagrangian (1.6) is

$$L = - \left(\frac{2M_{\text{Pl}}^2}{N} + \frac{\alpha \dot{\Phi}^2}{N^3} \right) Q + \left(\frac{\varepsilon \dot{\Phi}^2}{N} - 2N\Lambda \right) a_1 a_2 a_3, \quad (\text{A.5})$$

with

$$Q = a_1 \dot{a}_2 \dot{a}_3 + a_2 \dot{a}_1 \dot{a}_3 + a_3 \dot{a}_1 \dot{a}_2. \quad (\text{A.6})$$

Varying L with respect to the lapse N gives the first order constraint,

$$\mathcal{C} = \left(\frac{2M_{\text{Pl}}^2}{N^2} + \frac{3\alpha \dot{\Phi}^2}{N^4} \right) Q - \left(\frac{\varepsilon \dot{\Phi}^2}{N^2} + 2\Lambda \right) a_1 a_2 a_3 = 0, \quad (\text{A.7})$$

whereas varying with respect to a_k and Φ gives the second order equations. Let us set $N = 1$ and assume the configuration to be almost isotropic,

$$a_k = a + \delta a_k, \quad \Phi = \int \psi dt + \delta \Phi.$$

Then $Q = 3a\dot{a}^2 + \delta Q + \dots \equiv Q_0 + \delta Q + \dots$ with

$$\delta Q = 2a\dot{a}(\delta\dot{a}_1 + \delta\dot{a}_2 + \delta\dot{a}_3) + \dot{a}^2(\delta a_1 + \delta a_2 + \delta a_3), \quad (\text{A.8})$$

and $\mathcal{C} = (2M_{\text{Pl}}^2 + 3\alpha\psi^2)3a\dot{a}^2 - (\varepsilon\psi^2 + 2\Lambda)a^3 + \delta\mathcal{C} + \dots \equiv \mathcal{C}_0 + \delta\mathcal{C} + \dots$ with

$$\delta\mathcal{C} = (2M_{\text{Pl}}^2 + 3\alpha\psi^2)\delta Q + 2(9\alpha a\dot{a}^2 - \varepsilon a^3)\psi\delta\dot{\Phi} - (\varepsilon\dot{\Phi}^2 + 2\Lambda)a^2(\delta a_1 + \delta a_2 + \delta a_3), \quad (\text{A.9})$$

while the Lagrangian (A.5) expands as $L = L_0 + \frac{1}{2} \delta^2 L + \dots$ where $\delta^2 L$ is a quadratic in $\delta\dot{\Phi}$, $\delta\dot{a}_k$, δa_k form. The first order condition $\delta\mathcal{C} = 0$ can be used to express $\delta\dot{\Phi}$ in terms of the other variables,

$$\delta\dot{\Phi} = \mathcal{W}(\delta\dot{a}_1 + \delta\dot{a}_2 + \delta\dot{a}_3) + \dots, \quad (\text{A.10})$$

where the dots denote terms without derivatives and

$$\mathcal{W} = \frac{(2M_{\text{Pl}}^2 + 3\alpha\psi^2)H}{(\varepsilon - 9\alpha H^2)a\psi}. \quad (\text{A.11})$$

Inserting (A.10) to $\delta^2 L$ gives a quadratic form containing only $\delta\dot{a}_k$ and δa_k ,

$$\delta^2 L = A(\delta\dot{a}_1 + \delta\dot{a}_2 + \delta\dot{a}_3)^2 + 2B(\delta\dot{a}_1\delta\dot{a}_2 + \delta\dot{a}_1\delta\dot{a}_3 + \delta\dot{a}_2\delta\dot{a}_3) + \dots, \quad (\text{A.12})$$

where

$$A = (\varepsilon - 3\alpha H^2)a^3\mathcal{W}^2 - 4\alpha a^2 H\mathcal{W}, \quad B = -\frac{1}{2}(2M_{\text{Pl}}^2 + \alpha\psi^2). \quad (\text{A.13})$$

The form (A.12) has eigenvalues

$$\lambda_1 = 3A + 2B, \quad \lambda_2 = \lambda_3 = -B, \quad (\text{A.14})$$

which should be non-negative. Therefore, using the above values of A, B, \mathcal{W} and requiring the kinetic term to be positive, one arrives at the no-ghost conditions

$$\lambda_1 = \frac{a(18\alpha H^2\psi^2 + 6M_{\text{Pl}}^2 H^2 - \varepsilon\psi^2)(9\alpha^2 H^2\psi^2 - 6\alpha M_{\text{Pl}}^2 H^2 + \varepsilon(\alpha\psi^2 + 2M_{\text{Pl}}^2))}{\psi^2(9\alpha H^2 - \varepsilon)^2} > 0 \quad (\text{A.15})$$

and

$$\lambda_2 = \lambda_3 = \frac{a}{2}(2M_{\text{Pl}}^2 + \alpha\psi^2) > 0. \quad (\text{A.16})$$

It turns out that these are, respectively, the same conditions as those for the generic scalar and tensor perturbations.

Let us now see what happens if $K \neq 0$. One notes first that the scalar sector condition (A.15) can be obtained in a simpler way. Indeed, restricting to only isotropic perturbations, $\delta a_1 = \delta a_2 = \delta a_3 = \delta a$, (A.12) reduces to

$$\delta^2 L = 3(3A + 2B)\delta\dot{a}^2 = 3\lambda_1\delta\dot{a}^2 + \dots, \quad (\text{A.17})$$

hence λ_1 can be computed by perturbing only Φ and the scale factor a of the isotropic background. This can be easily done for non-zero values of the spatial curvature K . Let us consider the FLRW metric for an arbitrary K ,

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right]. \quad (\text{A.18})$$

Inserting this to (1.6) yields the reduced Lagrangian

$$L = -\frac{3\alpha a}{N^3}\dot{\Phi}^2(\dot{a}^2 + KN^2) + \frac{6M_{\text{Pl}}^2 a}{N}(KN^2 - \dot{a}^2) + \left(\frac{\varepsilon}{N}\dot{\Phi}^2 - 2\Lambda N\right)a^3 \quad (\text{A.19})$$

and the constraint $\mathcal{C} = \partial L / \partial N$. Setting $N = 1$, perturbing the scale factor $a \rightarrow a + \delta a$ and the scalar field $\Phi \rightarrow \Phi + \delta\Phi$, expanding L and \mathcal{C} with respect to $\delta a, \delta\Phi$ and then using $\delta\mathcal{C} = 0$ to express $\delta\dot{\Phi}$ in terms of $\delta\dot{a}$, one obtains

$$\delta^2 L = 3\lambda_1\delta\dot{a}^2 + \dots \quad (\text{A.20})$$

with

$$\lambda_1 = \frac{a(18\alpha H^2 \psi^2 + 6M_{\text{Pl}}^2 H^2 - \epsilon \psi^2)(9\alpha^2 H^2 \psi^2 - 6\alpha M_{\text{Pl}}^2 H^2 + \epsilon(\alpha \psi^2 + 2M_{\text{Pl}}^2))}{\psi^2 (9\alpha H^2 - \epsilon)^2}, \quad (\text{A.21})$$

where

$$\epsilon = \varepsilon - \frac{3\alpha K}{a^2}. \quad (\text{A.22})$$

This reduces to λ_1 in (A.15) if $K = 0$. The ghost in the scalar sector will be absent if $\lambda_1 > 0$.

One may wonder why the no-ghost condition depends on K . Normally this is not the case, since changing K changes the spatial curvature but not the temporal derivatives of the metric, hence the metric kinetic term does not depend on K . However, the kinetic term of the scalar field contains the non-minimal contribution $G_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi$, and this depends on K , since the Einstein tensor does. Therefore, the full kinetic term of the theory depends on K , which is why the scalar sector no-ghost condition is K -dependent. At the same time, the no-ghost condition in the tensor sector is expected to be the same for any K and to be given by (A.16), because $\delta\Phi = 0$ for the tensor (and vector) perturbations.

The above conclusions are confirmed by analysing generic perturbations of the isotropic background. The corresponding equations for the perturbations are shown in Appendix C below (for $K = 0$). Inserting these equations to (A.3) and integrating by parts to get rid of the second derivatives determines the second variation $\delta^2 L$, which splits into three independent parts corresponding to contributions of the scalar, vector and tensor modes. The kinetic part of $\delta^2 L$ in each sector is a manifestly positive definite quadratic form multiplied by a factor depending on the background amplitudes. The positivity of these factors for the tensor and vector modes requires of λ_2 defined by Eq.(A.16) is positive, while the positivity in the scalar sector requires that λ_1 given by (A.21),(A.22) is positive.

The conclusion is that ghosts will be absent if λ_1 and λ_2 defined by Eqs.(A.16), (A.21), (A.22) are positive. Since ψ is unbounded, the condition (A.16), $2M_{\text{Pl}}^2 + \alpha\psi^2 > 0$, requires that $\alpha > 0$, which explains Eq.(2.22) in the main text. Eqs.(A.21), (A.22) are equivalent to Eq.(2.23) in the main text.

B Stability of flat space in the full F4

For the sake of completeness, we derive here the no-ghost conditions for flat space in the full F4 theory. Choosing the metric in the FLRW form (A.18) and $\Phi = \Phi(t)$, the Lagrangian (1.3) becomes (up to a total derivative)

$$L_{\text{F4}} = L_J + L_P + L_G + L_R - 2\Lambda N a^3, \quad (\text{B.1})$$

with

$$\begin{aligned}
L_J &= \frac{3aV_J}{N^3} \dot{\Phi}^2 (\dot{a}^2 + KN^2), \\
L_P &= -\frac{3\dot{a}V_P}{N^5} \dot{\Phi}^3 (\dot{a}^2 + KN^2), \\
L_G &= \frac{6a}{N} \left(V_G (KN^2 - \dot{a}^2) - V'_G \dot{\Phi} a \dot{a} \right), \\
L_R &= -\frac{8\dot{a}\dot{\Phi}V'_R}{N^3} (\dot{a}^2 + 3KN^2).
\end{aligned} \tag{B.2}$$

Varying this with respect to N, a, Φ gives the constraint and the field equations. Whatever the functions $V_P(\Phi), V_J(\Phi), V_G(\Phi), V_R(\Phi)$ are (unless $V_P = V_J = V'_G = 0$), the equations admit the flat space solution

$$N = 1, \quad K = -1, \quad a = t, \tag{B.3}$$

with the scalar field determined by

$$V_P \dot{\Phi}^3 - t V_J \dot{\Phi}^2 + t^2 V'_G \dot{\Phi} = \frac{\Lambda}{3} t^3. \tag{B.4}$$

The no-ghost conditions for this solution can be obtained by applying the described above procedure: perturbing the $K = -1$ metric to compute the kinetic term for the scalar modes, and considering the anisotropic deformations of the $K = 0$ metric to describe the tensor modes. This shows that the scalar and tensor ghosts will be absent if the following two conditions hold,

$$\begin{aligned}
9V_P \psi^3 - 5tV_J \psi^2 + (2t^2V'_G + 8V'_R)\psi + 2tV_G &> 0, \\
3V_P \psi^3 - tV_J \psi^2 + 8V'_R \psi + 2tV_G &> 0,
\end{aligned} \tag{B.5}$$

where $\psi = \dot{\Phi}$. Although these conditions are fulfilled if $V_P = V_R = 0, V_J = -\alpha < 0$, and $V_G = M_{\text{Pl}}^2$, as chosen in the main text, in general they impose non-trivial conditions on the coefficient functions $V_J(\Phi), \dots, V_R(\Phi)$. Therefore, flat space within the full F4 theory can be unstable and it will be ghost-free if only the coefficients V_J, \dots, V_R are properly chosen.

C Equations for generic perturbations

Consider a homogeneous and isotropic background with $K = 0$,

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -dt^2 + a^2(t) (d\mathbf{x})^2, \quad \Phi^{(0)} = \int \psi(t) dt. \tag{C.1}$$

Its generic inhomogeneous and anisotropic perturbations can be expressed in the synchronous gauge as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \quad \Phi = \int \psi(t) dt + \delta\Phi, \tag{C.2}$$

where

$$\delta g_{0\mu} = 0, \quad \delta g_{ik} = 2a^2(t) h_{ik}(t) e^{i\mathbf{p}\mathbf{x}}, \quad \delta\Phi = \phi(t) e^{i\mathbf{p}\mathbf{x}}. \quad (\text{C.3})$$

One can assume the momentum vector to be oriented along the third axis, $\mathbf{p} = (0, 0, p)$. The h_{ik} tensor can be decomposed as

$$h_{ik}(t) = \sum_{m=1}^6 R_m(t) h_{ik}^{(m)} \quad (\text{C.4})$$

where the basis matrices

$$\begin{aligned} h_{ik}^{(1)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & h_{ik}^{(2)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, & h_{ik}^{(3)} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ h_{ik}^{(4)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & h_{ik}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & h_{ik}^{(6)} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{C.5})$$

Here $h_{ik}^{(1)}$ and $h_{ik}^{(2)}$ are diagonal and have a non-zero overlap with the \mathbf{p} vector, they correspond to the scalar modes; $h_{ik}^{(3)}$ and $h_{ik}^{(4)}$ are traceless but have a non-zero overlap with \mathbf{p} and describe the vector modes; $h_{ik}^{(5)}$ and $h_{ik}^{(6)}$ are traceless and have no overlap with \mathbf{p} , hence they describe the TT-tensor modes.

Injecting this into the linearized field equations (A.2), the variables separate, giving an independent set of equations for the scalar amplitudes $R_1(t), R_2(t), \phi(t)$, another independent set for the vector amplitudes $R_3(t), R_4(t)$, and finally equations for the tensor amplitudes $R_5(t), R_6(t)$. The scalar field perturbation $\phi(t)$ contributes only to the scalar sector.

The amplitudes R_m are dimensionless but $\phi, \psi, a, \mathbf{p}$ and the time t have dimensions. One therefore passes to dimensionless variables as described by Eqs.(2.17), (2.20), (2.21) in the main text, so that $a = a_0 a$, etc. One also defines the dimensionless time $\tau = H_0 t$, and from now on the dot will denote $d/d\tau$. The dimensionless Hubble parameter

$$h = \frac{\dot{a}}{a} \equiv \frac{1}{a} \frac{da}{d\tau} \quad (\text{C.6})$$

is related to y defined by Eq.(2.17) in the main text via $h^2 = y$. One sets $p = a_0 H_0 P$ and

$$\psi = -\sqrt{\frac{2\rho_{\text{cr}}\Omega_6}{3\alpha H_0^2}} \Psi, \quad \phi = -\frac{1}{H_0} \sqrt{\frac{2\rho_{\text{cr}}\Omega_6}{3\alpha H_0^2}} \phi. \quad (\text{C.7})$$

As a result, the independent equations for the scalar sector amplitudes $w(\tau) \equiv$

$R_1(\tau)$, $u(\tau) \equiv R_2(\tau)$, and $\phi(\tau)$ read

$$3\Omega_6\Psi(\zeta - 3h^2)\dot{\phi} - 3h(3\Omega_6\Psi^2 + 1)\dot{w} - \frac{P^2}{a^2} \{(\Omega_6\Psi^2 + 1)(w + u) + 2\Omega_6 h\Psi\phi\} = 0, \quad (\text{C.8})$$

$$(\Omega_6\Psi^2 + 1)(\dot{w} + \dot{u}) + 2\Omega_6 h\dot{\phi} + 3\Omega_6\Psi(\zeta - h^2)\phi = 0, \quad (\text{C.9})$$

$$\begin{aligned} & (\zeta - h^2)\ddot{\phi} - 2h\Psi\ddot{w} + \left(3(\zeta - 3h^2)\Psi - 2h\dot{\Psi} - 2\Psi\dot{h}\right)\dot{w} \\ & + (3\zeta - 2\dot{h} - 3h^2)h\dot{\phi} + [3(\zeta - h^2)\dot{\Psi} + 9(\zeta - h^2)h\Psi]w \\ & - \frac{2P^2}{3a^2}(\dot{\Psi} + h\Psi)(w + u) - \frac{P^2}{3a^2}(2\dot{h} + 3h^2 - 3\eta)\phi = 0. \end{aligned} \quad (\text{C.10})$$

Here the first two equations are first order in $d/d\tau$ because they correspond to the δE_{00} and δE_{03} gravitational constraints. The third equation is second order and corresponds to δE_{Φ} . The scalar sector contains also second order equations δE_{11} , δE_{22} , and δE_{33} , but these are not independent and follow from (C.8)–(C.10) in view of the Bianchi identities.

Equations in the tensor sector are much simpler. Each of the two tensor amplitudes $w(\tau) \equiv R_5(\tau)$ and $u(\tau) \equiv R_6(\tau)$ fulfills exactly the same equation

$$\begin{aligned} & (\Omega_6\Psi^2 + 1)\ddot{w} + \left(2\Omega_6\Psi\dot{\Psi} + 3(\Omega_6\Psi^2 + 1)h\right)\dot{w} \\ & - \left(2(\Omega_6\Psi^2 + 1)(2\dot{h} + 3h^2) + 2\Omega_6(3\zeta\Psi^2 + 4h\Psi\dot{\Psi}) - 6\Omega_0 + \frac{P^2}{a^2}(\Omega_6\Psi^2 - 1)\right)w = 0. \end{aligned} \quad (\text{C.11})$$

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